4.5.1 Direct Methods

The linear equations given by Eq. (4.5.4) have a block tridiagonal structure and can be written in vector-matrix form given by Eq. (4.4.29). However, in this case, due to slightly different notation, we write Eq. (4.4.29) again,

$$\mathbb{A}\mathbf{U} = \mathbf{F} \tag{4.5.6}$$

Here $\mathbb{A}$ a denotes the coefficient matrix same as that defined in Eq. (4.4.30), but with different indices,

$$\mathbb{A} = \begin{vmatrix} A_1 & C_1 & & & & & \\ B_2 & A_2 & C_2 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & B_j & A_j & C_j & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & B_{J-1} & A_{J-1} & C_{J-1} \\ & & & & & B_J & A_J \end{vmatrix} \tag{4.5.7}$$

and with A_j, B_j and C_j denoting I-dimensional matrices and I_I denoting the identity matrix of order I

$$A_j = \begin{vmatrix} 1 & -\theta_x & & & & & \\ -\theta_x & 1 & -\theta_x & & & & \\ & -\theta_x & 1 & -\theta_x & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -\theta_x & 1 & -\theta_x \\ & & & & & -\theta_x & 1 \end{vmatrix} \tag{4.5.8a}$$

$$B_j = C_j = -\theta_y I_I \tag{4.5.8b}$$

In Eq. (4.5.6), $\mathbf{U}$ and $\mathbf{F}$ are 3-dimensional compound vectors (i.e., vectors whose components are I-dimensional vectors) and are defined by

$$\mathbf{U} = \begin{vmatrix} \underset{\sim}{u}_1 \\ \underset{\sim}{u}_2 \\ \vdots \\ \underset{\sim}{u}_j \\ \vdots \\ \underset{\sim}{u}_J \end{vmatrix}, \quad \underset{\sim}{u}_j = \begin{vmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{i,j} \\ \vdots \\ u_{I,j} \end{vmatrix}, \quad \mathbf{F} = \begin{vmatrix} \underset{\sim}{F}_1 \\ \underset{\sim}{F}_2 \\ \vdots \\ \underset{\sim}{F}_j \\ \vdots \\ \underset{\sim}{F}_J \end{vmatrix}, \tag{4.5.9}$$

where

$$\begin{aligned} \underset{\sim}{F}_1 &= \underset{\sim}{f}_1 + \theta_x \underset{\sim}{w}_1 + \theta_y \underset{\sim}{u}_0 \\ \underset{\sim}{F}_j &= \underset{\sim}{f}_j + \theta_x \underset{\sim}{w}_j \qquad 2 \le j \le J-1 \\ \underset{\sim}{F}_J &= \underset{\sim}{f}_J + \theta_x \underset{\sim}{w}_J + \theta_y \underset{\sim}{u}_{J+1} \end{aligned} \tag{4.5.10}$$

$$\underset{\sim}{f}_j = \begin{vmatrix} F_{1,j} \\ F_{2,j} \\ \vdots \\ F_{I,j} \end{vmatrix}, \quad \underset{\sim}{w}_j = \begin{vmatrix} u_{0,j} \\ 0 \\ \vdots \\ 0 \\ u_{I+1,j} \end{vmatrix}, \quad \underset{\sim}{u}_0 = \begin{vmatrix} u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{I,0} \end{vmatrix}, \quad \underset{\sim}{u}_{J+1} = \begin{vmatrix} u_{1,J+1} \\ u_{2,J+1} \\ \vdots \\ u_{I,J+1} \end{vmatrix} \tag{4.5.11}$$

In Eq. (4.5.11) $\underset{\sim}{f}_j$ comes from the right-hand side of Eq. (4.5.4b); once the function $f(x, y)$ in the Poisson equation is given, $\underset{\sim}{f}_j$ is known. The column vector $\underset{\sim}{w}_j$ represents the boundary conditions at $i = 0$ and $i = I + 1$, and $\underset{\sim}{u}_0$ and $\underset{\sim}{u}_{J+1}$ represent the boundary conditions at $j = 0$ and $j = J + 1$, respectively. Note that $w_{i,j} = 0$ for $2 \leq i \leq I - 1$, $w_{1,j} = u_{0,j}$, and $w_{I,j} = u_{I+1,j}$. Of course, all of the $u_{i,j}$ which enter into Eq. (4.5.10) are known quantities determined from the boundary conditions.

In some problems all of the boundary conditions may not be given in terms of u, but are given some in terms of its derivatives. In those cases, the structure of the A_j, B_j, C_j matrices in the coefficient matrix $\mathbb{A}$ can change. To illustrate, consider Problem 4.5 with the boundary conditions are of the form

$$i = 0, \quad \frac{\partial u}{\partial x} = 0; \quad i = I + 1, \quad u = 0 \tag{4.5.12a}$$

$$j = 0, \quad \frac{\partial u}{\partial y} = 0; \quad j = J + 1, \quad u = 0 \tag{4.5.12b}$$

The boundary conditions at $i = 0$ and $j = 0$ may be approximated to first order by the forward difference formula (4.3.9) by

$$u_0 - u_1 = 0 \tag{4.5.13}$$

or to second order, requiring

$$u(\zeta) = a_0 + a_1\zeta + a_2\zeta^2$$

to satisfy $u(o) = u_0$, $u(\Delta\zeta) = u_1$ and $u(2\Delta\zeta) = u_2$, and then setting $du(0)/d\zeta = a_1 = 0$. This procedure yields

$$u_0 - \frac{4}{3}u_1 + \frac{1}{3}u_2 = 0 \tag{4.5.14}$$

The choice given by Eq. (4.5.14) allows the boundary conditions at $i = 0$ and $j = 0$ to be written in the form

$$i = 0, \quad u_{0,j} - \frac{4}{3}u_{1,j} + \frac{1}{3}u_{2,j} = 0 \quad 1 \leq j \leq J \tag{4.5.15a}$$

$$j = 0, \quad u_{i,0} - \frac{4}{3}u_{i,1} + \frac{1}{3}u_{i,2} = 0 \quad 1 \leq i \leq I \tag{4.5.15b}$$

For $i = 1$, Eq. (4.5.4b) becomes

$$u_{1,j} - \theta_x(u_{2,j} + u_{0,j}) - \theta_y(u_{1,j+1} + u_{1,j-1}) = -\delta^2 f_{1,j} = F_{1,j}$$

and, with Eq. (4.5.15a), it can be written as

$$u_{1,j} - \theta_x(u_{2,j} + \frac{4}{3}u_{1,j} - \frac{1}{3}u_{2,j}) - \theta_y(u_{1,j+1} + u_{1,j-1}) = F_{1,j}$$

or as

$$\left(1 - \frac{4}{3}\theta_x\right) u_{1,j} - \frac{2}{3}\theta_x u_{2,j} - \theta_y(u_{1,j+1} + u_{1,j-1}) = F_{1,j}, \quad 2 \le j \le J-1 \quad (4.5.16a)$$

For $i = I$, Eq. (4.5.4b) becomes

$$u_{I,j} - \theta_x(u_{I+1,j} + u_{I-1,j}) - \theta_y(u_{I,j+1} + u_{I,j-1}) = F_{I,j}$$

and with the boundary condition at $i = I + 1$, that is,

$$u_{I+1,j} = 0 \quad (4.5.15c)$$

Eq. (4.5.4b) at $i = I$ can be written as

$$u_{I,j} - \theta_x u_{I-1,j} - \theta_y(u_{I,j+1} + u_{I,j-1}) = F_{I,j} \quad 2 \le j \le J - 1 \quad (4.5.16b)$$

For $j = 1$, Eq. (4.5.4b) becomes

$$u_{i,1} - \theta_x(u_{i+1,1} + u_{i-1,1}) - \theta_y(u_{i,2} + u_{i,0}) = F_{i,1}$$

and, with Eq. (4.5.15b), can be written as

$$u_{i,1} - \theta_x(u_{i+1,1} + u_{i-1,1}) - \theta_y(u_{i,2} + \frac{4}{3}u_{i,1} - \frac{1}{3}u_{i,2}) = F_{i,1}$$

or as

$$\left(1 - \frac{4}{3}\theta_y\right) u_{i,1} - \theta_x(u_{i+1,1} + u_{i-1,1}) - \frac{2}{3}\theta_y u_{i,2} = F_{i,1}, \quad 2 \le i \le I-1 \quad (4.5.16c)$$

For $j = J$, Eq. (4.5.4b) becomes

$$u_{i,J} - \theta_x(u_{i+1,J} + u_{i-1,J}) - \theta_y(u_{i,J+1} + u_{i,J-1}) = F_{i,J}$$

and with the boundary condition at $j = J + 1$, that is,

$$u_{i,J+1} = 0 \quad (4.5.15d)$$

Eq. (4.5.4b) at $j = J$ can be written as

$$u_{i,J} - \theta_x(u_{i+1,J} + u_{i-1,J}) - \theta_y(u_{i,J-1}) = F_{i,J}, \quad 2 \le i \le I - 1 \quad (4.5.16d)$$

At $i = j = 1$, Eq. (4.5.4b), with the relations given by Eqs. (4.5.15a,b), becomes

$$\left(1 - \frac{4}{3}\theta_x - \frac{4}{3}\theta_y\right) u_{1,1} - \frac{2}{3}\theta_x u_{2,1} - \frac{2}{3}\theta_y u_{1,2} = F_{1,1} \quad (4.5.17a)$$

At $i = 1$, $j = J$, Eq. (4.5.4b), with the relations given by Eqs. (4.5.15a,d) becomes

$$\left(1 - \frac{4}{3}\theta_x\right) u_{1,J} - \frac{2}{3}\theta_x u_{2,J} - \theta_y u_{1,J-1} = F_{1,J} \tag{4.5.17b}$$

At $i = I$, $j = 1$, Eq. (4.5.4b), with the relations given by Eqs. (4.5.15b,c) becomes

$$\left(1 - \frac{4}{3}\theta_y\right) u_{I,1} - \theta_x u_{I-1,1} - \frac{2}{3}\theta_y u_{I,2} = F_{I,1} \tag{4.5.17c}$$

At $i = I$, $j = J$, Eq. (4.5.4b), with the relations given by Eqs. (4.5.15c,d) becomes

$$u_{I,J} - \theta_x u_{I-1,J} - \theta_y u_{I,J-1} = F_{I,J} \tag{4.5.17d}$$

The matrices A_j, B_j and C_j in the coefficient matrix $\mathbb{A}$ become

$$A_1 = \begin{vmatrix} a_1^* & -\frac{2}{3}\theta_x & & & & & \\ -\theta_x & a_2^* & -\theta_x & & & & \\ & -\theta_x & a_2^* & -\theta_x & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -\theta_x & a_2^* & -\theta_x \\ & & & & & -\theta_x & a_2^* \end{vmatrix} \tag{4.5.18a}$$

$$A_j = \begin{vmatrix} a_3^* & -\frac{2}{3}\theta_x & & & & & \\ -\theta_x & 1 & -\theta_x & & & & \\ & -\theta_x & 1 & -\theta_x & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -\theta_x & 1 & -\theta_x \\ & & & & & -\theta_x & 1 \end{vmatrix} \qquad 2 \le j \le J-1 \tag{4.5.18b}$$

$$A_J = \begin{vmatrix} a_3^* & -\frac{2}{3}\theta_x & & & & & \\ -\theta_x & 1 & -\theta_x & & & & \\ & -\theta_x & 1 & -\theta_x & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & -\theta_x & 1 & -\theta_x \\ & & & & & -\theta_x & 1 \end{vmatrix} \tag{4.5.18c}$$

$$B_j = -\theta_y I_I \qquad 2 \le j \le J \tag{4.5.18d}$$

$$C_1 = -\tfrac{2}{3}\theta_y I_I \tag{4.5.18e}$$

$$C_j = -\theta_y I_I \qquad 2 \le j \le J-1 \tag{4.5.18f}$$

where

$$a_1^* = 1 - \frac{4}{3}\theta_x - \frac{4}{3}\theta_y, \quad a_2^* = 1 - \frac{4}{3}\theta_y, \quad a_3^* = 1 - \frac{4}{3}\theta_x \tag{4.5.19}$$

Whether the boundary conditions are given in terms of u or its derivatives, the solution of Eq. (4.5.6) can be obtained by the block-elimination method as discussed in subsection 4.4.3 or Gauss' elimination method discussed below. Since the coefficient matrix $\mathbb{A}$ has large blocks of zero elements, it is more efficient to solve Eq. (4.5.6) with the block-elimination method than with the Gaussian elimination method. As we shall see shortly, however, it is still necessary to make partial use of the Gaussian elimination in the block elimination for the solution of elliptic equations.

The block elimination method for this problem is identical to the one discussed before except for the difference of the indices in the coefficient matrix, Eq. (4.4.30). So for convenience the two steps in this method are repeated below.

In the first step of the forward sweep, Γ_j and Δ_j are computed from

$$\Delta_1 = A_1 \tag{4.5.20a}$$

$$\Gamma_j \Delta_{j-1} = B_j \qquad j = 2, 3, \ldots J \tag{4.5.20b}$$

$$\Delta_j = A_j - \Gamma_j C_{j-1} \qquad j = 2, 3, \ldots J \tag{4.5.20c}$$

In the second part of the forward sweep, the $\underset{\sim}{w}_j$ are computed from

$$\underset{\sim}{w}_1 = \underset{\sim}{F}_1 \tag{4.5.21a}$$

$$\underset{\sim}{w}_j = \underset{\sim}{F}_j - \Gamma_j \underset{\sim}{w}_{j-1} \qquad 2 \le j \le J \tag{4.5.21b}$$

In the backward sweep, the $\underset{\sim}{u}_j$ are computed from

$$\Delta_J \underset{\sim}{u}_J = \underset{\sim}{w}_J \tag{4.5.22a}$$

$$\Delta_j \underset{\sim}{u}_j = \underset{\sim}{w}_j - C_j \underset{\sim}{u}_{j+1} \qquad j = J-1, J-2, \ldots, 1 \tag{4.5.22b}$$

In the application of the block elimination method to solve the Laplace difference equations, the Δ_j matrix in Eqs. (4.5.20b) and (4.5.22) is a full matrix of order I and is not a tridiagonal matrix except for $J = 1$. Thus, the inversion of Δ_j is not a trivial task. On the other hand, in the application of this method to solve the difference equations for boundary layers (Chapter 7), the order of Δ_j matrix is generally small. For this reason, the inversion of the Δ_j matrix is relatively simple.

To solve Eqs. (4.5.20b) and (4.5.22), we use the Gaussian elimination method and write both equations in the form

$$A\underset{\sim}{x} = \underset{\sim}{b} \tag{4.5.23}$$

Here $A \equiv [a_{ij}]$ is a square matrix of order n, that is,

$$A \equiv \begin{vmatrix} a_{11} & a_{12} & \cdot & a_{1j} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i1} & a_{i2} & \cdot & a_{ij} & \cdot & \cdot & a_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & a_{nj} & \cdot & \cdot & a_{nn} \end{vmatrix} \tag{4.5.24}$$

and $\underset{\sim}{x} = (x_1, \ldots, x_i, \ldots, x_n)^T$ and $\underset{\sim}{b} = (b_1, \ldots, b_i, \ldots, b_n)^T$ with T denoting the transpose. According to the Gaussian elimination method, the elements of $\underset{\sim}{x}$ are given by

$$x_i = \frac{1}{a_{ii}^{(i-1)}} \left[b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j \right], \quad i = n, \ldots, 1 \tag{4.5.25}$$

where

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}, \qquad \begin{aligned} k &= 1, \ldots, n-1 \\ j &= k+1, \ldots, n \\ i &= k+1, \ldots, n \\ a_{ij}^{(0)} &= a_{ij} \end{aligned} \tag{4.5.26a}$$

$$b_i^{(k)} = b_i^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)}, \qquad \begin{aligned} k &= 1, \ldots, n-1 \\ i &= k+1, \ldots, n \\ b_i^{(0)} &= b_i \end{aligned} \tag{4.5.26b}$$

Table 4.2 gives the FORTRAN listing based an the Gaussian elimination. Thus, the block-elimination method together with the Gaussian elimination method

Table 4.2. FORTRAN Listing of Subroutine GAUSS

```
      SUBROUTINE GAUSS (N,M,A,B)
      DIMENSION A(100,100),B(100,100)
      DO 100  K = 1,N-1
      KP      = K + 1
      DO 100  I = KP,N
      R       = A(I,K)/A(K,K)
      DO 200  J = KP,N
200   A(I,J)  = A(I,J) - R*A(K,J)
      DO 100  J = 1,M
100   B(I,J)  = B(I,J) - R*B(K,J)
      DO 300  K = 1,M
      B(N,K)  = B(N,X)/A(N,N)
      DO 300  I = N-1,1,-1
      IP      = I + 1
      DO 400  J = IP,N
400   B(I,K)  = B(I,K) - A(I,J)*B(J,K)
300   B(I,K)  = B(I,K)/A(I,I)
      RETURN
      END
```

can be used to solve Eq. (4.5.6). A listing of a subroutine for this purpose is given in Table 4.3 for A_j, B_j and C_j matrices given by Eqs. (4.5.8a,b).

To use the subroutine in Table 4.3, the number of grid points in the x and y directions must be specified by I ($\equiv$ II) and J ($\equiv$ JJ), respectively, the coefficients θ_x ($\equiv$ TX), θ_y ($\equiv$ TY) in Eq. (4.5.4a), and the compound vector $\mathbf{F}$ ($\equiv$ F) on the right-hand side of Eq (4.5.6). The compound vector $\mathbf{F}$ is obtained from Eq (4.5.10) once the forcing function [$f(x, y)$ in Eqs. (4.5.1)] is defined and the boundary conditions on the four sides of the rectangle are given.

Example 4.5. Compute the temperature distribution in a square region of sides unity subject to the following boundary conditions

$$T(x,0) = T(x,1) = 0, \quad T(0,y) = \sin \pi y \quad \text{and} \quad T(1,y) = e^{\pi} \sin \pi y$$

by solving the heat condition equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad 0 \le x,\ y \le 1$$

Compare your solutions with the analytical solution at $x = 0.2$, 0.5 and 0.9.

$$T(x,y) = e^{\pi x} \sin \pi y$$

Take $\Delta x = \Delta y = 1/10$.

Solution. Table E4.10 presents a comparison between the numerical and analytical results at $x = 0.2$, 0.5 and 0.9 as a function of y. Appendix A contains the computer program.

Table E4.10. Comparison of FDS and AS

	x=0.2		x=0.5		x=0.9	
y	FDS	AS	FDS	AS	FDS	AS
0.1	0.58693	0.57924	1.504	1.48652	5.23614	5.22301
0.2	1.1164	1.10178	2.86078	2.62753	9.95973	9.93476
0.3	1.53659	1.51647	3.93753	3.89176	13.70839	13.67403
0.4	1.80637	1.78271	4.62884	4.57504	6.11517	16.07478
0.5	1.89933	1.87446	4.86705	4.81048	16.9445	16.90203
0.6	1.80637	1.78271	4.62884	4.57504	16.11517	16.07478
0.7	1.53659	1.51647	3.93753	3.89176	13.70838	13.67403
0.8	1.1164	1.10178	2.86078	2.82753	9.95972	9.93476
0.9	0.58693	0.57924	1.504	1.48652	5.23614	5.22301

4.5.2 Iterative Methods

Iterative solutions which may be based on *point* or *block* iterations are more popular than the direct methods used to solve the Laplace difference equations. Again the large number of zero elements in the coefficient matrix A greatly reduces the computational effort required in each iteration. However, care must